The More, the Less, and the Much More: An Introduction to Łukasiewicz logic, Part 1

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Every Bureaucrat's Dream: Number-Crunching beats Reasoning

The answer to the

Ultimate Question of Life, The Universe, and Everything

from the supercomputer, Deep Thought, specially built for this purpose. It took Deep Thought $7\frac{1}{2}$ million years to compute and check the answer.

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Here it is:

$$42$$

After D. Adams, *The Hitchhiker's Guide to the Galaxy*, 1978

DIDATTICA

- Attrattività
- Sostenibilità
- Stage
- Mobilità Internazionale
- Borse di studio
- Dispersione
- Efficacia
- Soddisfazione
- Occupazione

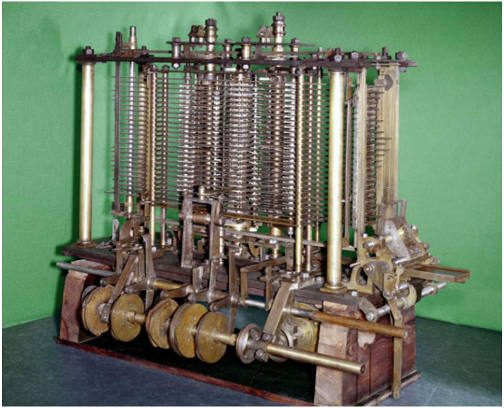
RICERCA

- Fondi esterni
- Ricerca
- Alta formazione

POSIZIONE	ATENEO	PUNTI
1	Verona	84
2	Trento	84
3	Politecnico di Milano	79
4	Bologna	78
5	Padova	76
6	Politecnica delle Marche	75
7	Venezia Ca' Foscari	73
8	Milano Bicocca	73
9	Siena	73
10	Politecnico di Torino	73
11	Pavia	72
12	Piemonte Orientale	71
13	Milano Statale	70
14	Ferrara	68
15	Udine	66
16	Macerata	65
17	Firenze	63
18	Viterbo	62
19	Modena e Reggio Emilia	61
20	Venezia Iuav	60
21	Torino	59
22	Roma Foro Italico	58
23	Salerno	58
24	Pisa	56

Ada Lovelace, 1815 – 1852

Ada is looking forward to move to Italy to enrol in some Italian university. She is well known to enjoy number-crunching, but also — and perhaps even more — logic and reasoning. How can she best take advantage of the newspaper's ranking?



Babbage's Analytical Engine, 1834–1871

Aware that one person's number-crunching may be another's nonsense, she decides to go for logic and reasoning instead. She starts writing down her *desiderata*.

Ada's ideal university U should satisfy the following.

- U enjoys substantial international mobility.
- U has considerable reputation for graduate education.
- U invests much more in research than in undergraduate teaching.
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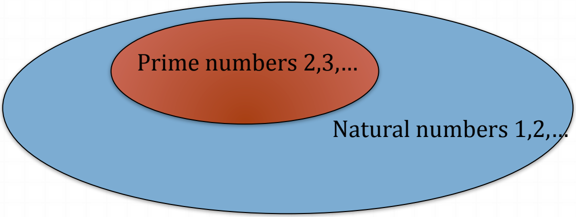
It seems to Ada that Italy is beyond reason indeed.

Precise *vs.* Vague Predicates



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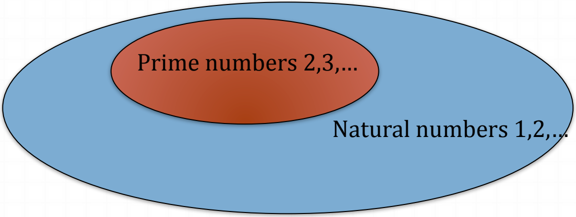

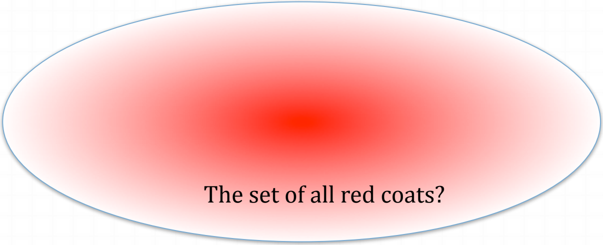

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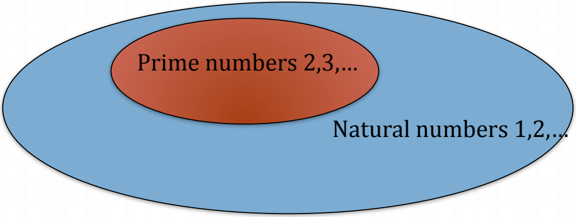
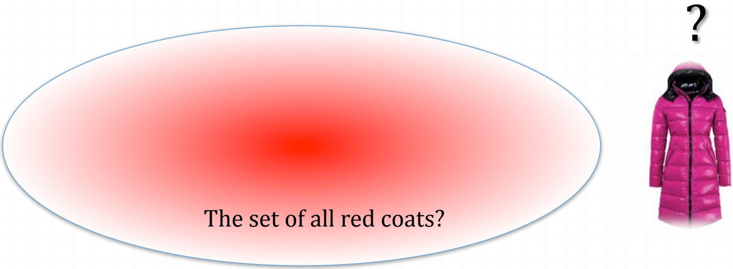
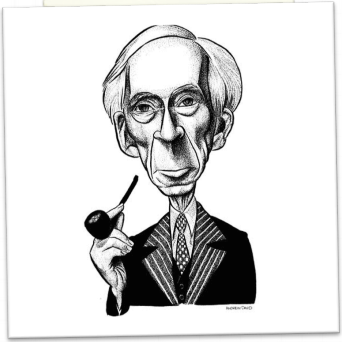






Bertrand Russell, 1872–1970

`Late(x)`, like `Red(x)` or `Tall(x)`, is a vague (monadic) predicate. Instantiations such as `Tall(VM)` yield vague propositions. Vague predicates have been given much attention by the analytic philosophers, beginning with a paper by Bertrand Russell in the Twenties. The subject is nowadays known as *Theories of Vagueness*.

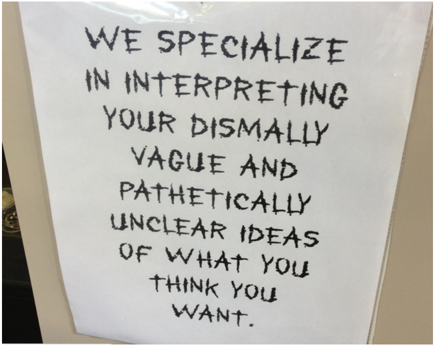
Theory-neutral features of vagueness

Features of a (monadic) vague predicate *R*:

(FV1) R admits *borderline cases* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by (a term naming a constant) $c \in D$ such that it is unclear whether $R(c)$ holds or its negation $\neg R(c)$ does.

(FV2) R lacks *sharp boundaries* over the intended domain of interpretation D, *i.e.* there is no clearly defined boundary separating the extension of $R(\cdot)$ from its anti-extension.

(FV3) R is susceptible to a *Sorites series* over the intended domain of interpretation D, *i.e.* there are instantiations of $R(x)$ by $c_1, \ldots, c_n \in D$ such that it is clear that $R(c_1)$ holds, it is clear that $R(c_n)$ does not hold, and it seems at least plausible that if $R(c_i)$ holds then so does $R(c_{i+1})$, for each $i \in \{1, \ldots, n-1\}$.



Shop sign

This tutorial is devoted to the modest aim of solving Ada's problem. For this, we are going to develop the (propositional) logic of (certain) vague predicates, concentrating on reasoning rather than number-crunching.

Clear Assumptions about Vague Predicates

Assumption I

Each vague predicate has a well-defined extension.

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The assumption does not entail that the predicate is precise, or that it does not admit borderline cases, etc. Indeed, given any x, it is a matter of classical logic that:

- Either it is the case that $\mathsf{Tall}(x)$, i.e. x is a clear, indisputable case of a tall individual;
- Or it is not the case that $\mathsf{Tall}(x)$, i.e. x is not a clear, indisputable case of a tall individual.

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Consequently, one cannot assert a vague predicate tentatively, or to a degree.

In the *Begriffsschrift*, Frege introduced the sign $\vdash$ as a compound formation:

$-$	the *content* stroke
$\mid$	the *judgement* stroke
$\vdash$	the *assertion* sign
$\vdash \alpha$	means: α (assertion of).

Hence, by

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Comment. There are formal systems, such as Pavelka's logic, where inference is indexed by a degree. But it is unclear whether one can make sense at all of the idea of "asserting (or assuming) a proposition to a degree", and even less of the idea of "deducing α from β to a degree".

Assumption II

We only consider vague predicates admitting an antonym.

E.g., Tall–Short, Near–Far, etc.

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Assumptions I & II directly lead to 3 notions of negation:

Predicate	Extension
—Tall	Set-theoretic complement of the extension of Tall
¬Tall	Extension of the opposite predicate Short
∼Tall	Extension of the predicate non-Tall

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Assumption III

We only consider the negation connective $\neg$.

Tall and Red are fundamentally different vague predicates.

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- Tall has a natural antonymic, or opposite, or contrary predicate, namely, Short. In symbols,

$$\neg\mathsf{Tall}(x) \equiv (\neg\mathsf{Tall})(x) \equiv \mathsf{Short}(x).$$

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- Red does not have a natural contrary. There is no name for opposite-to-Red in the colour spectrum. Similarly: Cute, Nice, etc. Hence:

$\neg$Red just doesn't make sense.

The distinction above is strictly a matter of logic, not of linguistic usage or what have you.

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- The extension of —Tall is the set of individuals which are not a clear, indisputable case of tallness.
- Hence, the extension of —(—Tall) coincides with the extension of Tall: set-theoretic complement is idempotent.

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- The extension of $\neg$Tall is the set of individuals which are a clear, indisputable case of shortness.

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- The extension of $\neg$Tall is the set of individuals which are a clear, indisputable case of shortness.
- Hence, the extension of $\neg(\neg\text{Tall})$ coincides with the extension of Tall: the antonym of the antonym of Tall is Tall.

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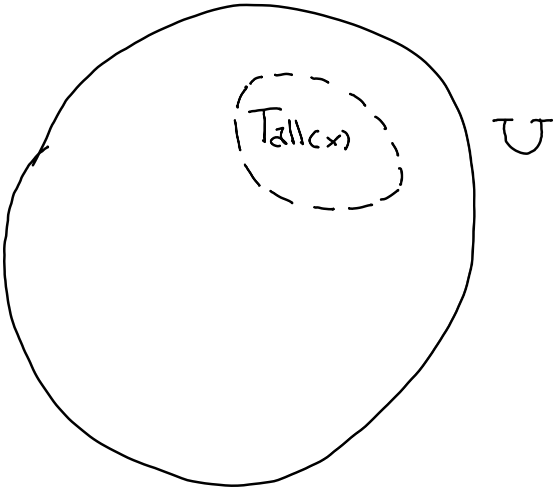
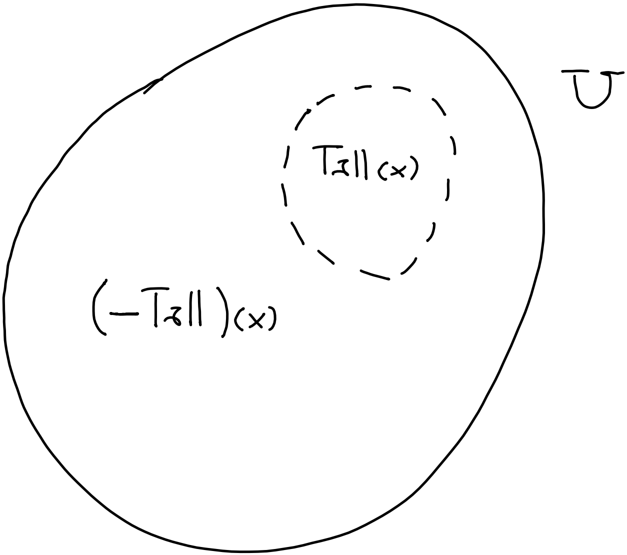
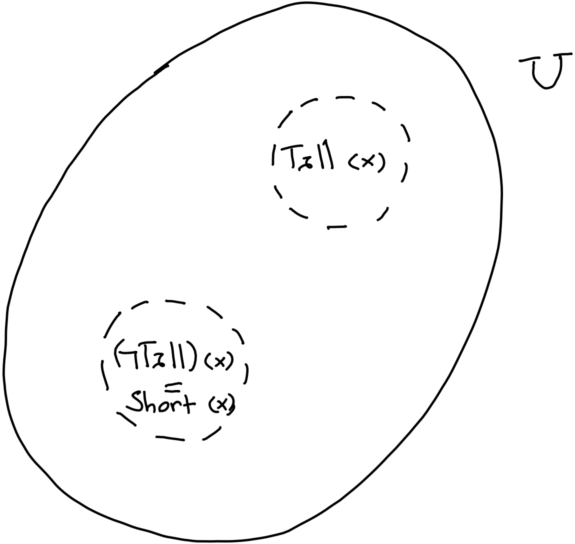
The negation $\sim$ must fail the Double Negation Law

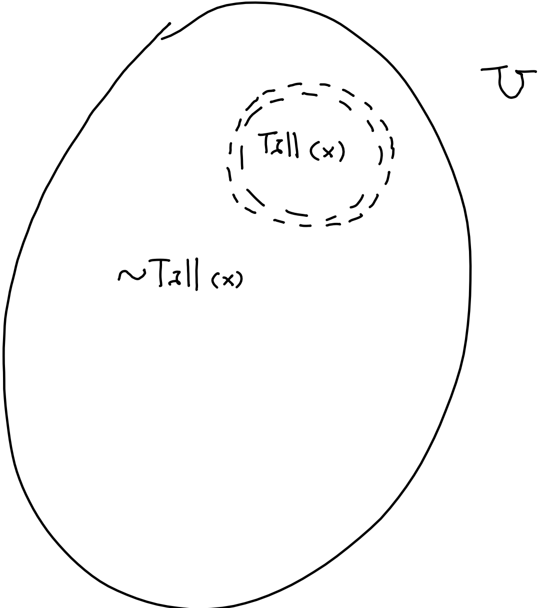
Indeed, $\sim$ behaves like an Intuitionistic pseudo-complement:

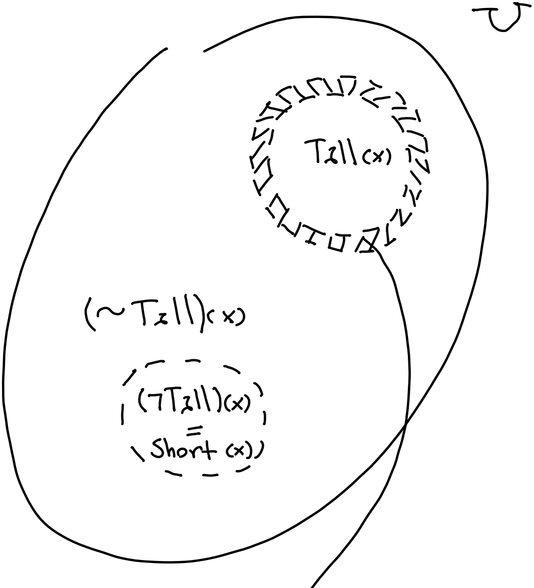
- The extension of Red is the set of objects which are a clear, indisputable case of redness.
- The extension of $\sim$Red is the set of objects which are a clear, indisputable case of non-redness.
- Hence, the extension of $\sim(\sim$Red$)$ is the set of objects which do not qualify as a clear case of non-redness; but in general they will not qualify as a clear case of redness, either.


Tall(x)
U


Tall(x)
(−Tall)(x)
U


U
Tall (x)
(¬Tall) (x)
=
Short (x)


U
Tall (x)
~Tall (x)


U
Tall(x)
(~Tall)(x)
(¬Tall)(x)
=
Short(x)

Let us take stock:

- $\neg$Tall applies to anything that is clearly opposite to tall, i.e. is clearly short.
- $\neg$Red just doesn't make sense, because there is no opposite to redness.

We henceforth restrict attention to predicates such as Tall, which admit of antonyms such as $\neg\mathsf{Tall} \equiv \mathsf{Short}$. We only consider the negation $\neg$.

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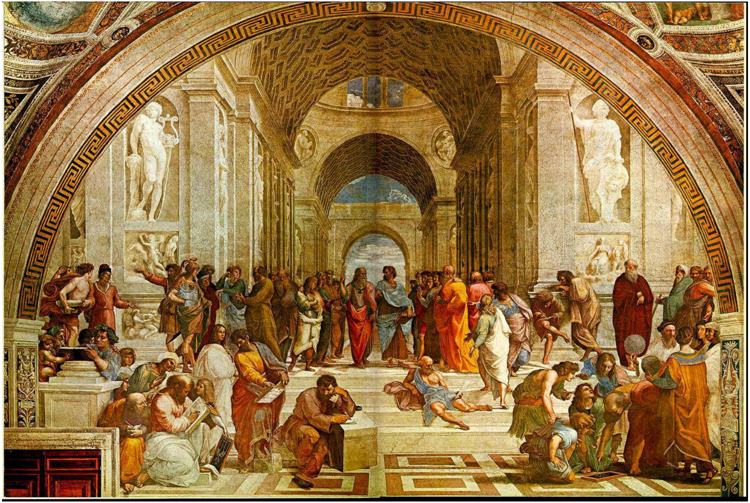
We have made some assumptions about a unary connective, negation. The next key issue now is:

What binary connectives are basic for vague predicates?

True, Truer, Much Truer



Raffaello Sanzio, La Scuola di Atene, *ca. 1509.*

E. Casari, *Comparative logic*, Synthese, 1987.

ETTORE CASARI

COMPARATIVE LOGICS

1. INTRODUCTION

Comparative Logic was created by Aristotle at the very beginnings of logic. In the *Topics* he developed, in particular, a highly satisfactory theory of the nine kinds of propositions which arise by crossing comparisons of majority, minority and equality (μᾶλλον, ἧττον, ὁμοίως) with situations in which 'one is said of two (ἑνὸς περὶ δύο λεγομένου)', 'two are said of one (δυοῖν περὶ ἑνὸς λεγομένων)' and 'two are said of two (δυοῖν περὶ δύο λεγομένων)', i.e.,

x is more *A* than *y*;
x is less *A* than *y*;
x is as much *A* as *y*;
x is more *A* than *B*;
x is less *A* than *B*;
x is as much *A* as *B*;
x is more *A* than *y* is *B*;
x is less *A* than *y* is *B*;

Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives*.

P1. x is more T than z.

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For instance:

P1. Ada is more tall than Carolina.

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P1. Ada is more tall than Carolina.

P2. Blaise is more tall than Carolina.

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P3. Ada is more (more tall than Carolina) than (that by which Blaise is more tall than Carolina).

C. Ada is more tall than Blaise.

Note that as we step up from the subsentential to the sentential level, to model the Aristotelian

$$x \text{ is more } T \text{ than } z$$

with a *single connective independent of* T, it seems unavoidable to move on to the sentence

$$T(x) \text{ is more true than } T(z).$$

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Key Fact

The comparison connective

$$\alpha \text{ is more true than } \beta$$

produces a classical proposition out of the given α and β.

Hence the connective is more true than cannot play a fundamental rôle in the logic of vague predicates.

A Standard Mistake

Many-valued logics (after Hájek) are logics of comparative truth wherein the implication connective

$$\alpha \to \beta$$

is read

$$\alpha \text{ is less true than } \beta.$$

This is simply untenable.

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Proper Reformulation

Many-valued logics (after Hájek) are logics of comparative truth wherein the assertion

$$\vdash \alpha \to \beta$$

is read

It is the case that α is less true than β.

Standard account of meaning of a proposition/predicate as its truth conditions:

> [...] *to grasp a thought is to know the conditions for it to be true.*
>
> M. Dummett, 1976

E.g., you know what $\mathsf{Prime}(x)$ means as soon as you can tell a prime number when you see it. Compare:

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Just Wrong ($\star$)

You know what Tall(x) means as soon as you can tell a (clearly, indisputably) tall person when you see one.

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You know what $\mathsf{Tall}(x)$ means as soon as you can tell a (clearly, indisputably) tall person when you see one.

For, what about a tallish, though not indisputably tall person? And even an indisputably short person? You may perfectly meet $(\star)$ and yet be completely in the dark as to whether a clear, indisputable case of a short person indeed is short. That's no grasping of $\mathsf{Tall}(x)$, on any sensible account.

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Example

In Intuitionistic logic, the Lindenbaum-Tarski equivalence class of any proposition α is uniquely determined by the collection of (intuitionistic) valuations that make α true.

(Mathematically, this is precisely why in Intuitionistic logic, like in classical logic, one can develop Stone-Esakia-Priestley duality for Heyting algebras in the extensional language of *clopen upper sets*, and give up functions, i.e. Fregean "courses of values".)

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Let us take stock: Only in Gödel-Dummett logic can the implication be plainly read as "less true than". Cf. S. Aguzzoli, VM, *Two principles in many-valued logic*, dedicated to Petr Hájek, forthcoming.

So what can we resort to if "more true than" won't do? Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives.*

P1. x is more T than z.

P2. y is more T than z.

P3. x is more (more T than z) than (that by which y is more T than z).

C. x is more T than y.

A propositional translation of Aristotle's example:

P1. $T(x)$ is more true than $T(z)$.

P2. $T(y)$ is more true than $T(z)$.

P3. ($T(x)$ is more true than $T(z)$) is more true than ($T(y)$ is more true than $T(z)$).

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A propositional translation of Aristotle's example:

P1. $T(x)$ is more true than $T(z)$.

P2. $T(y)$ is more true than $T(z)$.

P3. ($T(x)$ is much more true than $T(z)$) is more true than ($T(y)$ is much more true than $T(z)$).

So what can we resort to if "more true than" won't do? Aristotle's example, in the *Topics*, of inference with *comparatives of comparatives*.

P1. x is more T than z.

P2. y is more T than z.

P3. x is more (more T than z) than (that by which y is more T than z).

C. x is more T than y.

A propositional translation of Aristotle's example:

P1. $T(x)$ is more true than $T(z)$.

P2. $T(y)$ is more true than $T(z)$.

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For example,

$$\mathsf{Tall}(x) \rhd \mathsf{Young}(y)$$

means:

x is much more a case of tallness than y is a case of youth.

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Assumption IV

All vague propositions/predicates may be combined through $\rhd$ to yield new compound vague propositions/predicates.

In particular, this means that $\alpha \rhd \beta$ has again a well-defined extension, an antonym, and a well-defined anti-extension (Assumptions I–III).

Assumption V (Truth conditions of $\rhd$)

$$\vdash \alpha \rhd \beta \qquad \text{if, and only if,} \qquad \vdash \alpha \text{ and } \vdash \neg\beta.$$

Consider the sentence

Frege is much more intelligent than he is handsome.

This is a vague proposition of the from $\alpha \rhd \beta$. What does one mean when one asserts it, i.e. when

$$\vdash \alpha \rhd \beta \ ?$$

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$\vdash \alpha \triangleright \beta$ if, and only if, $\vdash \alpha$ and $\vdash \neg\beta$.

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Frege is much more intelligent than he is handsome.

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$$\vdash \alpha \triangleright \beta\ ?$$

Recalling our Assumption I, the only way to make logical sense of such an assertion is to interpret it as

Frege is intelligent, and Frege is ugly, or

$$\vdash \alpha \text{ and } \vdash \neg\beta.$$

Assumption V (Truth conditions of $\rhd$)

$$\vdash \alpha \rhd \beta \qquad \text{if, and only if,} \qquad \vdash \alpha \text{ and } \vdash \neg\beta.$$

Indeed, anything weaker than that will not attain assertoric force: it will necessarily be true to a non-maximal degree, which is incompatible with Assumption I.

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To see this, assume that in this world, Frege is actually full-on intelligent, and somewhat ugly, though not a clear, indisputable case of ugliness.

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To see this, assume that in this world, Frege is actually full-on intelligent, and somewhat ugly, though not a clear, indisputable case of ugliness.

Then there is a possible (=logically consistent) world wherein Frege is full-on intelligent, and full-on ugly: total ugliness coupled with total intelligence is not an inconsistent prospect. In this possible world, then, $\alpha \rhd \beta$ must be true to a higher degree than it is in the world we initially considered, whence $\alpha \rhd \beta$ could not have been full-on true there.

Our last assumption subsumes Assumption V:

Assumption VI (Course of Values of $\rhd$)

$\alpha \rhd \beta$ is the more true, the more α is truer than β.

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- The gap between the degree of truth of α and that of β; and
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This is the closest we get to the outright request that degrees of truth are magnitudes to be combined by arithmetic operations.

We are not quite asking that much, though. We are merely voicing the intuition that if, say, α is more true than β, then if the degree of truth of α grows while that of β stays constant, so does grow the degree of truth of $\alpha \rhd \beta$.

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$$\neg, \rhd, \top.$$

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$$\vdash \neg(\alpha \rhd \beta) \text{ if, and only if, } \alpha \text{ is no more true than } \beta.$$

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"Less true than"

$\vdash \neg(\alpha \rhd \beta)$ if, and only if, α is no more true than β.

For, if L.H.S. holds then it is full-on false that α is much more true than β. If we had "α more true than β" true to some degree, then we should have $\alpha \rhd \beta$ true, albeit possibly to a comparably small degree (Assumption VI). Hence "α no more true than β" holds.
Conversely, if "α no more true than β" holds, clearly $\alpha \rhd \beta$ is full-on false, hence $\vdash \neg(\alpha \rhd \beta)$.

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Hence the degree of truth of $(\star)$ is the minimum of the degree of truths of α and β.

Conjunction

We identify $\alpha \rhd (\alpha \rhd \beta)$ with the conjunction of α and β, written $\alpha \wedge \beta$. Observe: $\vdash \alpha \wedge \beta$ iff $\vdash \alpha$ and $\vdash \beta$.

Disjunction is defined through the De Morgan Laws.

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As an example of the foregoing:

Prelinearity

For any α and β we have:

$$\vdash \neg\left(\,(\alpha \rhd \beta) \wedge (\beta \rhd \alpha)\,\right).$$

This is a version of the standard *prelinearity axiom* in many-valued logic: $\vdash (\alpha \to \beta) \vee (\beta \to \alpha)$.

In the present version, it states the obvious: it is always full-on false that α is much more true than β, and at the same time β is much more true than α.

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We now have a language, and an intended semantics. It's time to talk about inference.

Axiomatisation

Jan Łukasiewicz, 1878–1956.

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Suppose α :="Ada is short", and β :="Ada is fat". Suppose further:

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That is, "Ada is short is less true than Ada is fat".

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Then we can infer: $\neg\alpha$, that is, “Ada is tall”.

Under our assumptions, this is a perfectly valid inference. It is no less grounded than a classical inference. It is a form of *modus tollens*:

$$\frac{\vdash \alpha \to \beta \qquad\qquad \vdash \neg\beta}{\vdash \neg\alpha} \quad (\textsc{mt})$$

Only deduction rule we use: *vague modus tollens*.

$$\frac{\vdash \neg(\alpha \rhd \beta) \qquad \vdash \neg\beta}{\vdash \neg\alpha} \quad (\text{VMT})$$

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Now we declare that a formula α in the language $\{\neg, \rhd, \top\}$ is provable if there exists a proof of α, that is, a finite sequence of formulæ $\alpha_1, \dots, \alpha_l$ a such that:

- $\alpha_l = \alpha$.
- Each α_i, $i < l$ is either an axiom, or is obtainable from α_j and α_k, $j, k < i$, via an application of *vague modus tollens*.

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So, what are the axioms?

$\boxed{\textit{Ex falso quodlibet}}$

$$\neg(\ \alpha \rhd \top\ )$$

$$\alpha \leqslant \top$$

Not much to say here: obvious.

$\boxed{\textit{A fortiori}}$

$$\alpha \rhd \beta \leqslant \alpha$$

For that by which α is truer than β cannot be less than the degree of truth of α itself. (In the extreme case, $\beta \equiv \bot$ and $\alpha \rhd \bot \equiv \alpha \leqslant \alpha$.)

$\boxed{\text{Transitivity of } \triangleright}$

$$(\gamma \triangleright \alpha) \triangleright (\gamma \triangleright \beta) \ \leqslant \ \beta \triangleright \alpha$$

This is best understood through a lengthy case analysis (3 propositions). It is to be thought of a consequence essentially of our crucial Assumption VI about correlation: $\gamma \triangleright \alpha$ and $\gamma \triangleright \beta$ compare in respect of truth value in the opposite manner as α compares to β, hence the R.H.S. has them reversed. For example, if α is more true than β, then R.H.S. is full-on false, so L.H.S. should be, too. And indeed, assuming γ is more true than both α and β, that by which γ is truer than α is smaller than that by which γ is truer than β.

$\boxed{\text{Contraposition}}$

$$\alpha \rhd \beta \;\leqslant\; \neg\beta \rhd \neg\alpha$$

Not much to say here. By our interpretation of negation and symmetry, think equality in place of $\leqslant$.

Conjunction is commutative

$$\alpha \rhd (\alpha \rhd \beta) \ \leqslant \ \beta \rhd (\beta \rhd \alpha)$$

Once we accept that L.H.S. is $\alpha \wedge \beta$, and hence R.H.S. is $\beta \wedge \alpha$, not much to say here: conjunction is obviously commutative (again, think equality in place of $\leqslant$).

Axiom system.

(A0) $\neg(\alpha \rhd \top)$ — *Ex falso quodlibet*

(A1) $\alpha \rhd \beta \leqslant \alpha$ — *A fortiori*

(A2) $(\gamma \rhd \alpha) \rhd (\gamma \rhd \beta) \leqslant \beta \rhd \alpha$ — Transitivity of $\rhd$

(A3) $\alpha \rhd (\alpha \rhd \beta) \leqslant \beta \rhd (\beta \rhd \alpha)$ — Conjunction is commutative

(A4) $\alpha \rhd \beta \leqslant \neg\beta \rhd \neg\alpha$ — Contraposition

$$\alpha \wedge \beta \equiv \alpha \rhd (\alpha \rhd \beta)$$

Deduction rule.

(R1) $\frac{\alpha \leqslant \beta \qquad \neg\beta}{\neg\alpha}$ — *Vague Modus Tollens.*

This Hilbert-style system defines Łukasiewicz logic.

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For example, our axiom defining conjunction standardly becomes

$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$

and now states that *disjunction* is commutative.

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We have:

$$\alpha \rhd \beta \equiv \neg(\alpha \rightarrow \beta).$$

The connective that I am denoting $\rhd$ is usually denoted $\ominus$.

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I am not just trying to be different: there are deep mathematical reasons why our "backwards" version is semantically more natural – cf. A. Pedrini's and D. McNeill's talks on Łukasiewicz and Intuitionism.

Intermezzo

Ada Lovelace, 1815 – 1852

By now Ada is beginning to feel a little more optimistic about the possibility of applying reasoning to her problem.

But while her list of *desiderata*:

- U enjoys substantial international mobility.
- U has considerable reputation for graduate education.
- U invests much more in research than in undergraduate teaching.
- ...

seems now less foreign to inference, Ada is still not clear about how she could compute a solution to her problem. Can she still use 0's and 1's, like her fellow countryman suggested? If not, then what should she attach to sentences in place of 0's and 1's?

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In the second and final part of this tutorial we will answer these questions, and we will eventually run a computer program that is able to compute a solution to Ada's problem, when properly fed her list of *desiderata*.

Thank you for your attention.